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Hierarchies Consistency Analysis for Nontransitive Problems

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Abstract—By assuming different hierarchies—or different hierarchy structures—for the same problem should yield the same aggregated performance index, an approach can be developed for evaluating the performance of multiple level and multiple criteria problems. This proposed approach has the advantage that it is completely based on a rational index and no requirement of transitivity is needed. Furthermore, the analyses of different criteria structures give further insights about the problem being considered. The disadvantage is, of course, we must assume that different hierarchies must yield the same aggregate performance index, which is not generally true for all problems under all conditions.

Keywords—Analytical hierarchy process, Transitivity property, Multiple criteria decision making, Hierarchies consistency analysis.

INTRODUCTION

To construct an effective performance evaluation model for a complex system forms the subject of intense research. Such an evaluation model not only provides an overall understanding of system performance, but also serves as a useful auditing tool for management purposes. Although a variety of mathematical and decision models have been proposed to address the problem, the most suitable model appears to be the multiple criteria type. One of the most useful approaches for performance analysis of the multiple criteria type is the analytical hierarchy process (AHP) due to Saaty [1,2].

A basic requirement of the AHP is that the criteria must have the transitivity property. In fact, this property is implied when pairwise comparison is performed. However, for many practical and complex problems, transitivity is a strong requirement and is frequently difficult to obtain. Another disadvantage is that the computational requirements for pairwise comparison goes up exponentially, and thus, for large and complex problems, this comparison procedure becomes unmanageable.

In this paper, we propose another approach—the hierarchies consistency analysis (HCA)—based on the fact that in the analysis of complex systems, the decision maker frequently wishes to emphasize a certain point or a certain combination of criteria. As a result of these different combinations, different hierarchy structures resulted. The HCA is based on these different hierarchies. An important assumption in the HCA approach is that since the elements in the bottom level of the hierarchies are the same, we can require the same overall performance index for the different hierarchies.

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A disadvantage of this assumption is that, in general, the overall performances for the different hierarchies may not all be the same. However, under certain conditions, the decision maker can construct different hierarchies whose overall performances are the same.

The most important advantage of the HCA is that it does not require any pairwise comparison in determining the weights of the criteria, and thus, no transitivity requirement is needed. A second advantage is due to the introduction of the rational index, which offers a natural and easy way to judge and to improve the assigned weights by the decision maker. By the use of the rational index and comparison between different hierarchies, a simpler procedure for calculating the weights of the criteria is formulated. Another advantage is that the analyses of these different structures give more insights to the problem being considered, hence it is useful to assist the decision maker to understand and to audit the performance status of the system from various different viewpoints.

THE CONCEPT

The fundamental concept of the hierarchies consistency analysis (HCA) is based on the fact that when a decision maker transfers a multiple criteria problem into hierarchy structure, several different hierarchy structures may be resulted depending on the desire of the management. However, since all these different structures or schema have the same components for the lowest level, the same aggregated performance index can be assumed under certain consistency conditions.

To illustrate the approach, let us consider the problem of constructing an evaluation model for teacher performance in higher education. The criteria to be considered are: the quality and quantity of teaching, research, and service. Two different hierarchy structures or schema can be constructed depending on whether the decision maker wants to separate quality with quantity.

HIERARCHY 1. The decision maker is concerned about the aggregated performance of the teacher in teaching, research, and service.

HIERARCHY 2. The decision maker is concerned about the quality and quantity of performances.

These two hierarchy structures are shown in Figure 1, where the assigned weights must obey the following equations:

$$
{}_1w_1^\circ + {}_1w_2^\circ + {}_1w_3^\circ = 1, \qquad {}_1w_{11}^\circ + {}_1w_{12}^\circ = 1, \qquad {}_1w_{21}^\circ + {}_1w_{22}^\circ = 1, \qquad {}_1w_{31}^\circ + {}_1w_{32}^\circ = 1,
$$
$$
{}_2w_{11}^\circ + {}_2w_{21}^\circ + {}_2w_{31}^\circ = 1, \qquad {}_2w_{12}^\circ + {}_2w_{22}^\circ + {}_2w_{32}^\circ = 1, \qquad {}_2w_1^\circ + {}_2w_2^\circ = 1.
$$

Consider the weight, ${}_iw_{jk}^\circ$, where the superscript indicates the number of iterations with $\circ$ indicating the initially assigned value or prior value, subscript i indicates the particular hierarchy structure, and subscripts j and k indicate the level and position in the hierarchy, respectively. For convenience, the initially assigned weights with superscript $\circ$ will be referred to as prior weights.

Since we have the same components in the lowest level of the hierarchy and the only difference between these two schema is the artificial structure, the same overall aggregated performance index can be assumed. In other words, we can assume

$$
\begin{array}{c} \text{Aggregated performance} \\ \text{index for} \\ \text{Hierarchy 1} \end{array} = \begin{array}{c} \text{Aggregated performance} \\ \text{index for} \\ \text{Hierarchy 2} \end{array} \tag{1}
$$

The above equation constitutes the basic idea of the approach. Based on equation (1), a rational criterion can be constructed which can be used to judge and to improve the process of weight assignment. Notice that no requirement of transitivity is needed.

Obviously, instead of only two hierarchy structures, more hierarchy structures can also be handled in the same way. In fact, more consistent analysis can be obtained if more different hierarchy structures can be constructed.

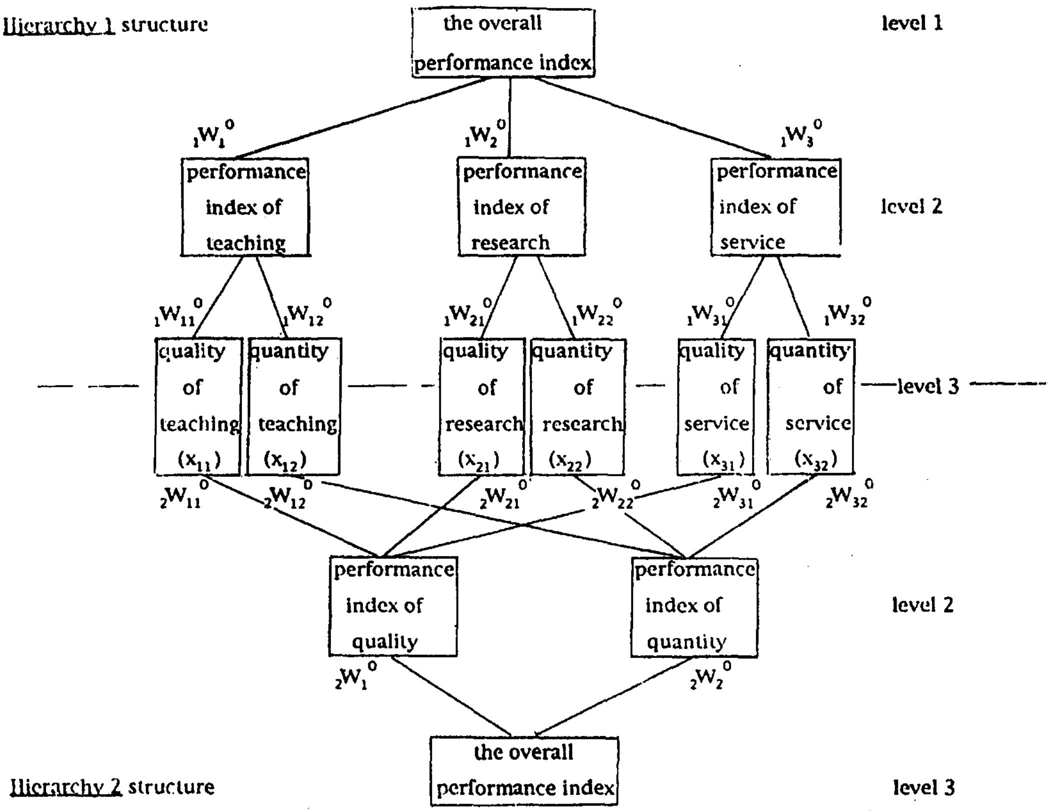

Figure 1. Hierarchy structure schema for faculty evaluation.

THE RATIONAL INDEX

One of the main problems in the analytical hierarchy approach is the process of assigning weights, where pairwise comparison and the eigenvalue approach were generally proposed. Although the consistence requirement between criteria appears reasonable. For complex problems, transitivity is not always possible. The HCA approach provides a convenient and simple method to analyze the weight assignment problem.

Let us consider the problem illustrated in Figure 1 again. The overall aggregated performance index for Hierarchy 1 is

$$ {}_1w_1^\circ \left({}_1w_{11}^\circ x_{11} + {}_1w_{12}^\circ x_{12}\right) + {}_1w_2^\circ \left({}_1w_{21}^\circ x_{21} + {}_1w_{22}^\circ x_{22}\right) + {}_1w_3^\circ \left({}_1w_{31}^\circ x_{31} + {}_1w_{32}^\circ x_{32}\right) $$

which can be rearranged to obtain

$$ \begin{aligned} &\left({}_1w_1^\circ\, {}_1w_{11}^\circ + {}_1w_2^\circ\, {}_1w_{21}^\circ + {}_1w_3^\circ\, {}_1w_{31}^\circ\right) \left(\frac{{}_1w_1^\circ\, {}_1w_{11}^\circ\, x_{11}}{A} + \frac{{}_1w_2^\circ\, {}_1w_{21}^\circ\, x_{21}}{A} + \frac{{}_1w_3^\circ\, {}_1w_{31}^\circ\, x_{31}}{A}\right) \\ &+ \left({}_1w_1^\circ\, {}_1w_{12}^\circ + {}_1w_2^\circ\, {}_1w_{22}^\circ + {}_1w_3^\circ\, {}_1w_{32}^\circ\right) \left(\frac{{}_1w_1^\circ\, {}_1w_{12}^\circ\, x_{12}}{B} + \frac{{}_1w_2^\circ\, {}_1w_{22}^\circ\, x_{22}}{B} + \frac{{}_1w_{31}^\circ\, {}_1w_{32}^\circ\, x_{32}}{B}\right), \end{aligned} $$

where $A = \left({}_1w_1^\circ\, {}_1w_{11}^\circ + {}_1w_2^\circ\, {}_1w_{21}^\circ + {}_1w_3^\circ\, {}_1w_{31}^\circ\right)$ and $B = \left({}_1w_1^\circ\, {}_1w_{12}^\circ + {}_1w_2^\circ\, {}_1w_{22}^\circ + {}_1w_3^\circ\, {}_1w_{32}^\circ\right)$.

Compare with the equation for Hierarchy 2:

$$ {}_2w_1^\circ \left({}_2w_{11}^\circ x_{11} + {}_2w_{21}^\circ x_{21} + {}_2w_{31}^\circ x_{31}\right) + {}_2w_2^\circ \left({}_2w_{12}^\circ x_{12} + {}_2w_{22}^\circ x_{22} + {}_2w_{32}^\circ x_{32}\right). $$

It is obvious that

$$ \begin{aligned} \Phi\left({}_2w_1^\circ\right) &= \left({}_1w_1^\circ\, {}_1w_{11}^\circ + {}_1w_2^\circ\, {}_1w_{21}^\circ + {}_1w_3^\circ\, {}_1w_{31}^\circ\right), \\ \Phi\left({}_2w_2^\circ\right) &= \left({}_1w_1^\circ\, {}_1w_{12}^\circ + {}_1w_2^\circ\, {}_1w_{22}^\circ + {}_1w_3^\circ\, {}_1w_{32}^\circ\right), \end{aligned} $$

$$\Phi\left({}_2w_{11}^{\circ}\right)=\frac{{}_1w_1^{\circ}\,{}_1w_{11}^{\circ}}{{}_1w_1^{\circ}\,{}_1w_{11}^{\circ}+{}_1w_2^{\circ}\,{}_1w_{21}^{\circ}+{}_1w_3^{\circ}\,{}_1w_{31}^{\circ}},$$
$$\Phi\left({}_2w_{12}^{\circ}\right)=\frac{{}_1w_1^{\circ}\,{}_1w_{12}^{\circ}}{{}_1w_1^{\circ}\,{}_1w_{12}^{\circ}+{}_1w_2^{\circ}\,{}_1w_{22}^{\circ}+{}_1w_3^{\circ}\,{}_1w_{32}^{\circ}},$$
$$\Phi\left({}_2w_{21}^{\circ}\right)=\frac{{}_1w_2^{\circ}\,{}_1w_{21}^{\circ}}{{}_1w_1^{\circ}\,{}_1w_{11}^{\circ}+{}_1w_2^{\circ}\,{}_1w_{21}^{\circ}+{}_1w_3^{\circ}\,{}_1w_{31}^{\circ}},$$
$$\Phi\left({}_2w_{22}^{\circ}\right)=\frac{{}_1w_2^{\circ}\,{}_1w_{22}^{\circ}}{{}_1w_1^{\circ}\,{}_1w_{12}^{\circ}+{}_1w_2^{\circ}\,{}_1w_{22}^{\circ}+{}_1w_3^{\circ}\,{}_1w_{32}^{\circ}},$$
$$\Phi\left({}_2w_{31}^{\circ}\right)=\frac{{}_1w_3^{\circ}\,{}_1w_{31}^{\circ}}{{}_1w_1^{\circ}\,{}_1w_{11}^{\circ}+{}_1w_2^{\circ}\,{}_1w_{21}^{\circ}+{}_1w_3^{\circ}\,{}_1w_{31}^{\circ}},$$
$$\Phi\left({}_2w_{32}^{\circ}\right)=\frac{{}_1w_3^{\circ}\,{}_1w_{32}^{\circ}}{{}_1w_1^{\circ}\,{}_1w_{12}^{\circ}+{}_1w_2^{\circ}\,{}_1w_{22}^{\circ}+{}_1w_3^{\circ}\,{}_1w_{32}^{\circ}},$$

where $\Phi({}_2w_j^{\circ})$ represents the calculated weights for Hierarchy 2 from the prior weights assigned by the decision maker for Hierarchy 1. Applying the same procedure, we can obtain the calculated or estimated weights for Hierarchy 1 from the assigned values by the decision maker for Hierarchy 2:

$$\Phi\left({}_1w_1^{\circ}\right)={}_2w_1^{\circ}\,{}_2w_{11}^{\circ}+{}_2w_{21}\,{}_1w_{12}^{\circ},$$
$$\Phi\left({}_1w_2^{\circ}\right)={}_2w_1^{\circ}\,{}_2w_{21}^{\circ}+{}_2w_2^{\circ}\,{}_1w_{22}^{\circ},$$
$$\Phi\left({}_1w_3^{\circ}\right)={}_2w_1^{\circ}\,{}_2w_{31}^{\circ}+{}_2w_2^{\circ}\,{}_1w_{32}^{\circ},$$
$$\Phi\left({}_1w_{11}^{\circ}\right)=\frac{{}_2w_1^{\circ}\,{}_2w_{11}^{\circ}}{{}_2w_1^{\circ}\,{}_2w_{11}^{\circ}+{}_2w_2^{\circ}\,{}_1w_{12}^{\circ}},$$
$$\Phi\left({}_1w_{12}^{\circ}\right)=\frac{{}_2w_2^{\circ}\,{}_1w_{12}^{\circ}}{{}_2w_1^{\circ}\,{}_2w_{11}^{\circ}+{}_2w_2^{\circ}\,{}_1w_{12}^{\circ}},$$
$$\Phi\left({}_1w_{21}^{\circ}\right)=\frac{{}_2w_1^{\circ}\,{}_2w_{21}^{\circ}}{{}_2w_1^{\circ}\,{}_2w_{21}^{\circ}+{}_2w_2^{\circ}\,{}_1w_{22}^{\circ}},$$
$$\Phi\left({}_1w_{22}^{\circ}\right)=\frac{{}_2w_2^{\circ}\,{}_1w_{22}^{\circ}}{{}_2w_1^{\circ}\,{}_2w_{21}^{\circ}+{}_2w_2^{\circ}\,{}_1w_{22}^{\circ}},$$
$$\Phi\left({}_1w_{31}^{\circ}\right)=\frac{{}_2w_1^{\circ}\,{}_2w_{31}^{\circ}}{{}_2w_1^{\circ}\,{}_2w_{31}^{\circ}+{}_2w_2^{\circ}\,{}_1w_{32}^{\circ}},$$
$$\Phi\left({}_1w_{32}^{\circ}\right)=\frac{{}_2w_2^{\circ}\,{}_1w_{32}^{\circ}}{{}_2w_1^{\circ}\,{}_2w_{31}^{\circ}+{}_2w_2^{\circ}\,{}_1w_{32}^{\circ}}.$$

Again, the weight ${}_iw_{jk}^{\circ}$ are the prior weights assigned by the decision maker and $\Phi({}_iw_{jk}^{\circ})$ are the calculated weights from the other hierarchy.

If we can assume that equation (1) is true and the weights assigned by the decision maker is reasonable, then the calculated values $\Phi({}_iw_{jk}^{\circ})$ should agree with the given values ${}_iw_{jk}^{\circ}$. However, in general, the decision maker may be unable to assign the correct weights, and thus, the calculated values may not agree with the given ones. The smaller this disagreement is, the more rational the decision maker is in assigning the weights. The HCA uses this disagreement to judge whether the decision maker is rational or not. Thus, we can formulate the following rational index (R.I.):

$$\text{R.I.}=\frac{\Sigma_j\left|{}_1w_j^{\circ}-\Phi\left({}_1w_j^{\circ}\right)\right|}{N_1}+\frac{\Sigma_j\left|{}_2w_j^{\circ}-\Phi\left({}_2w_j^{\circ}\right)\right|}{N_2},\tag{2}$$

where N_i is the number of prior weights assigned by the decision maker for Hierarchy 1. This rational index is a measure of whether the weights are assigned properly. If this index is nearly zero, then the weight assignment is nearly correct. But, on the other hand, if the index is very far removed from zero, then the weights need to be reassigned by the decision maker. This reassignment can be accomplished by the use of some kind average of the two aggregated indices as a guide for the decision maker. If the index is not too far removed from zero, or if it is within a certain tolerance, some kind of iterative procedures based on certain aggregated formula can be used to obtain the corrected weights.

Various different aggregate formulas in the literature can be used. One of the most straightforward formulas is simply taking the average between the assigned and the calculated weights. A more general one is the following convex combination which, obviously, includes the simple average when k equals to 1/2:

$$_1w_j^{n+1} = k_i w_j^n + (l-k)\phi\left({}_iw_j^n\right), \tag{3}$$

where superscript n represents the number of iterations, or equation (3) represents the n^{th} iteration, and k is a combination parameter $0 \le k \le 1$.

Since all the parameters in equation (3) are between 0 and 1 (i.e., $0 \le w \le 1$, $0 \le k \le 1$, $0 \le \phi(w) \le 1$), the above iteration formula forms a convergent series. In other words, this iterative process will eventually converge to the final desired weights.

EXAMPLE 1. This example has four criteria, x_{11}, x_{12}, x_{21}, and x_{22}, and two hierarchy structures. The assigned prior weights of criteria by the decision maker are listed in Figure 2. The computed weights obtained from the (prior) assigned weights are

$$\begin{aligned}
\Phi\left({}_1w_1^\circ\right) &= 0.6, & \Phi\left({}_2w_1^\circ\right) &= 0.660,\\
\Phi\left({}_1w_2^\circ\right) &= 0.4, & \Phi\left({}_2w_2^\circ\right) &= 0.340,\\
\Phi\left({}_1w_{11}^\circ\right) &= 0.8, & \Phi\left({}_2w_{11}^\circ\right) &= 0.364,\\
\Phi\left({}_1w_{12}^\circ\right) &= 0.2, & \Phi\left({}_2w_{12}^\circ\right) &= 0.470,\\
\Phi\left({}_1w_{21}^\circ\right) &= 0.3, & \Phi\left({}_2w_{21}^\circ\right) &= 0.636,\\
\Phi\left({}_1w_{22}^\circ\right) &= 0.7, & \Phi\left({}_2w_{22}^\circ\right) &= 0.530.
\end{aligned}$$

The rational index obtained is

$$\text{R.I.} = 0.48867.$$

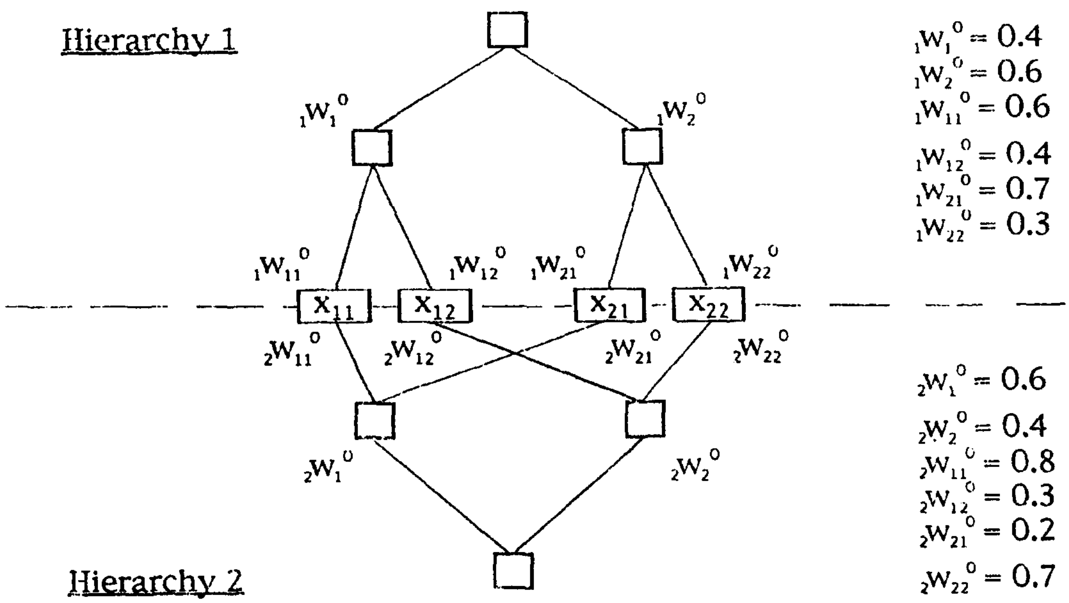

Figure 2. Hierarchy structure schema for Example 1.

Table 1. The compromise process of determining weights.

1st iter.		2nd iter.		3rd iter.	
${}_1w_1^o = 0.400$	$\phi({}_1w_1^o) = 0.600$	${}_1w_1^1 = 0.500$	$\phi({}_1w_1^1) = 0.509$	${}_1w_1^2 = 0.505$	$\phi({}_1w_1^2) = 0.495$
${}_1w_2^o = 0.600$	$\phi({}_1w_2^o) = 0.400$	${}_1w_2^1 = 0.500$	$\phi({}_1w_2^1) = 0.491$	${}_1w_2^2 = 0.505$	$\phi({}_1w_2^2) = 0.495$
${}_1w_{11}^o = 0.600$	$\phi({}_1w_{11}^o) = 0.400$	${}_1w_{11}^1 = 0.700$	$\phi({}_1w_{11}^1) = 0.720$	${}_1w_{11}^2 = 0.710$	$\phi({}_1w_{11}^2) = 0.710$
${}_1w_{12}^o = 0.400$	$\phi({}_1w_{12}^o) = 0.200$	${}_1w_{12}^1 = 0.300$	$\phi({}_1w_{12}^1) = 0.280$	${}_1w_{12}^2 = 0.290$	$\phi({}_1w_{12}^2) = 0.290$
${}_1w_{21}^o = 0.700$	$\phi({}_1w_{21}^o) = 0.300$	${}_1w_{21}^1 = 0.500$	$\phi({}_1w_{21}^1) = 0.536$	${}_1w_{21}^2 = 0.518$	$\phi({}_1w_{21}^2) = 0.518$
${}_1w_{22}^o = 0.300$	$\phi({}_1w_{22}^o) = 0.700$	${}_1w_{22}^1 = 0.500$	$\phi({}_1w_{22}^1) = 0.464$	${}_1w_{22}^2 = 0.482$	$\phi({}_1w_{22}^2) = 0.482$
${}_2w_1^o = 0.600$	$\phi({}_2w_1^o) = 0.660$	${}_2w_1^1 = 0.630$	$\phi({}_2w_1^1) = 0.600$	${}_2w_1^2 = 0.615$	$\phi({}_2w_1^2) = 0.615$
${}_2w_2^o = 0.400$	$\phi({}_2w_2^o) = 0.340$	${}_2w_2^1 = 0.370$	$\phi({}_2w_2^1) = 0.400$	${}_2w_2^2 = 0.385$	$\phi({}_2w_2^2) = 0.385$
${}_2w_{11}^o = 0.800$	$\phi({}_2w_{11}^o) = 0.364$	${}_2w_{11}^1 = 0.582$	$\phi({}_2w_{11}^1) = 0.583$	${}_2w_{11}^2 = 0.583$	$\phi({}_2w_{11}^2) = 0.583$
${}_2w_{12}^o = 0.300$	$\phi({}_2w_{12}^o) = 0.470$	${}_2w_{12}^1 = 0.385$	$\phi({}_2w_{12}^1) = 0.375$	${}_2w_{12}^2 = 0.380$	$\phi({}_2w_{12}^2) = 0.380$
${}_2w_{21}^o = 0.200$	$\phi({}_2w_{21}^o) = 0.364$	${}_2w_{21}^1 = 0.418$	$\phi({}_2w_{21}^1) = 0.417$	${}_2w_{21}^2 = 0.417$	$\phi({}_2w_{21}^2) = 0.417$
${}_2w_{22}^o = 0.700$	$\phi({}_2w_{22}^o) = 0.530$	${}_2w_{22}^1 = 0.615$	$\phi({}_2w_{22}^1) = 0.625$	${}_2w_{22}^2 = 0.620$	$\phi({}_2w_{22}^2) = 0.620$

This iterative process can be continued and the results are summarized in Table 1. After only three iterations, the process converged and the final rational weights are obtained.

EXAMPLE 2. This problem has three criteria and three hierarchy structures. The hierarchy structures, the prior and final weights of criteria are summarized in Figure 3 and Table 2. The final weights were obtained in 15 iterations. Since there are three hierarchies, the number of iterations required to reach convergence is higher than that of Example 1.

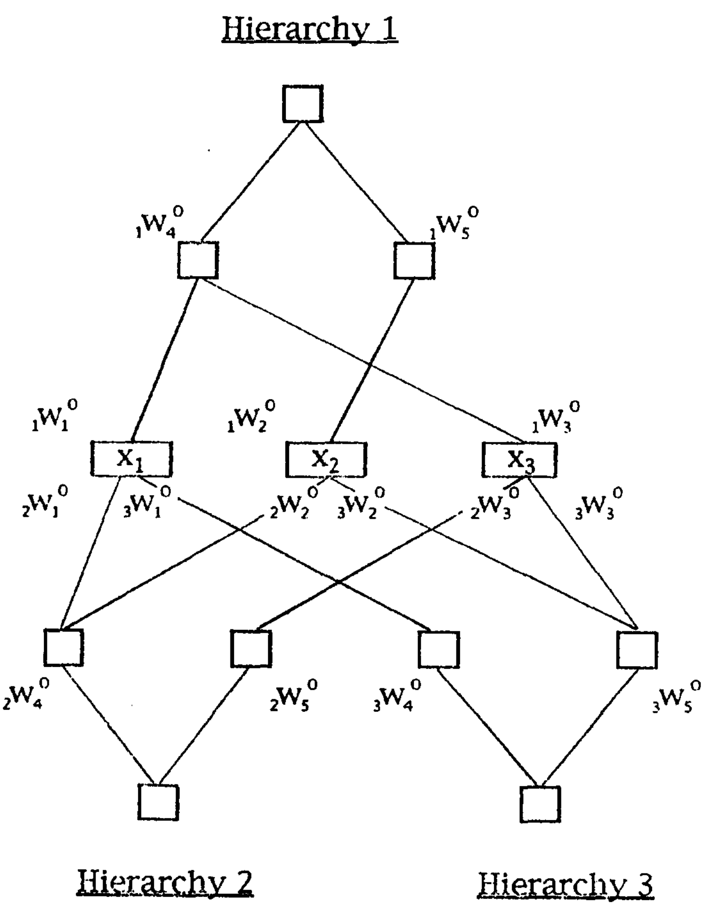

Figure 3. Hierarchy structure schema for Example 2.

Table 2. Prior and final weights of Example 2.

Prior weight	Final weight
${}_1w_1^o = 0.600$	$\phi\left({}_1w_1^{15}\right) = 0.3651$
${}_1w_2^o = 1.000$	$\phi\left({}_1w_2^{15}\right) = 1.0000$
${}_1w_3^o = 0.400$	$\phi\left({}_1w_3^{15}\right) = 0.6349$
${}_1w_4^o = 0.800$	$\phi\left({}_1w_4^{15}\right) = 0.8696$
${}_1w_5^o = 0.200$	$\phi\left({}_1w_5^{15}\right) = 0.1304$
${}_2w_1^o = 0.700$	$\phi\left({}_2w_1^{15}\right) = 0.7089$
${}_2w_2^o = 0.300$	$\phi\left({}_2w_2^{15}\right) = 0.2911$
${}_2w_3^o = 1.000$	$\phi\left({}_2w_3^{15}\right) = 1.0000$
${}_2w_4^o = 0.200$	$\phi\left({}_2w_4^{15}\right) = 0.4479$
${}_2w_5^o = 0.800$	$\phi\left({}_2w_5^{15}\right) = 0.5521$
${}_3w_1^o = 1.000$	$\phi\left({}_3w_1^{15}\right) = 1.0000$
${}_3w_2^o = 0.400$	$\phi\left({}_3w_2^{15}\right) = 0.1911$
${}_3w_3^o = 0.600$	$\phi\left({}_3w_3^{15}\right) = 0.8089$
${}_3w_4^o = 0.300$	$\phi\left({}_3w_4^{15}\right) = 0.3175$
${}_3w_5^o = 0.700$	$\phi\left({}_3w_5^{15}\right) = 0.6825$

EXAMPLE 3. FACULTY PERFORMANCE EVALUATION. The motivation for this work was originated in 1989 when Yuan-Ze Institute of Technology attempted to set up a set of performance evaluation model for its faculties. A committee was formed by ten representatives from the faculties. The first task of the committee was to decide what requirements would be included in building the performance evaluation model. After many rounds of discussions, the committee obtained the following conclusions.

1. The model should be a multiple criteria and multiple level model.
2. The method used to determine the weights of criteria should be simple but have a reasonable basis.
3. It is best to have a measure to decide whether the assigned criteria of weights are reasonable.
4. The model should provide more insights about the status of the faculty performances.

Based on these conclusions, the committee formed the hierarchy evaluation structure shown in the upper half of Figure 4. Initially, the AHP approach to determine the weights of criteria was used. But, it was soon discovered that AHP requires many pairwise comparisons. Thus, the committee decided to look into new approaches which can achieve the above desired conclusions.

To use the HCA approach, the committee formed two types of hierarchy structures: one is concerned with the performance of a faculty in teaching, research, and service—Hierarchy 1—and the other is concerned with the quality and quantity of performance of a faculty—Hierarchy 2. These structures are depicted in Figure 4.

After the committee assigned the prior weights for each criteria, the calculated weights are obtained. The R.I. obtained was 0.1379, which was acceptable by the committee. Further compromise calculations were also carried out by using the iteration process. The weights obtained are listed in Tables 3 and 4.

THE WEIGHTED GENERALIZED MEAN OPERATOR

Instead of equation (3), many other methods to aggregate or to combine the weights can be used. For example, the following weighted generalized mean operator, which was first proposed by Dujmovic and later by Dyckoff and Pedrycz [3], can be used:

$$f\left(x_1, x_2 \ldots, x_n,\ w_1, w_2 \ldots, w_n, p\right) = \left(w_1 x_1^p + w_2 x_2^p + \cdots + w_n x_n^p\right)^{1/p},$$

where w_1 represents the relative importance of the weights for the different criteria.

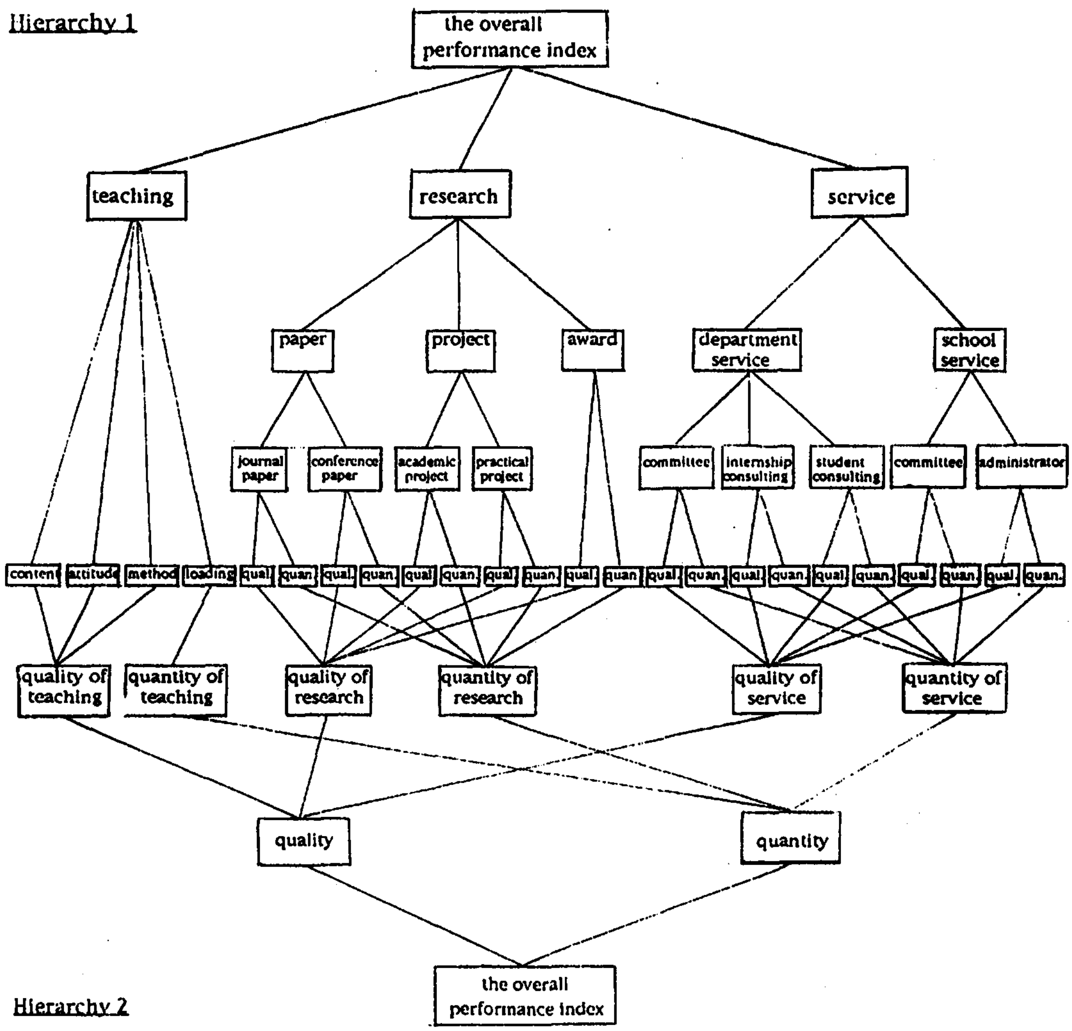

Figure 4. Faculty performance evaluation model.

By varying the parameter value p, the weighted generalized mean operator can produce various different aggregation operators. Some of the better known ones are:

$p = -\infty$, the minimum operator,
$p = -1$, the harmonic mean operator,
$p = 0$, the geometric mean operator,
$p = +1$, the arithmetic mean operator, and
$p = +\infty$, the maximum operator.

To illustrate the approach, consider Example 1 again. Using the numerical values listed in Figure 2 and Table 1, the generalized mean aggregation equation becomes

$$f = \left\{.505\left[(.710x_{11}^p + .290x_{12}^p)^{1/p}\right]^p + .95\left[(.518x_{21}^p + .482x_{22}^p)^{1/p}\right]^p\right\}^{1/p}.$$

If there are five sets of input data

$$x_1 = (.2,\ .4,\ .6,\ .8),$$
$$x_2 = (.4,\ .2,\ .8,\ .6),$$
$$x_3 = (.6,\ .4,\ .2,\ .8),$$
$$x_4 = (.8,\ .6,\ .2,\ .4),$$

Table 3. Prior and final weights for Hierarchy 1.

Criteria	Prior weight	Final weight
Teaching	0.4	0.398
content	0.3	0.239
attitude	0.3	0.239
method	0.2	0.224
loading	0.2	0.297
Research	0.4	0.402
paper	0.4	0.400
journal paper	0.7	0.600
quality	0.7	0.650
quantity	0.3	0.350
conference paper	0.3	0.400
quality	0.6	0.603
quantity	0.4	0.397
project	0.3	0.352
academic project	0.6	0.546
quality	0.6	0.599
quantity	0.4	0.401
practical project	0.4	0.454
quality	0.5	0.558
quantity	0.5	0.442
award	0.3	0.248
quality	0.8	0.708
quantity	0.2	0.293
Service	0.2	0.200
department service	0.5	0.500
committee	0.3	0.251
quality	0.6	0.594
quantity	0.4	0.406
internship consulting	0.3	0.350
quality	0.6	0.600
quantity	0.4	0.400
student consulting	0.4	0.399
quality	0.7	0.649
quantity	0.3	0.317
school service	0.5	0.500
committee	0.5	0.452
quality	0.6	0.596
quantity	0.4	0.404
administrator	0.5	0.548
quality	0.8	0.697
quantity	0.2	0.303

$$x_5 = (.4, .8, .2, .6),$$

then the aggregate performance index under different p values are listed in Table 5.

Consider Example 3 where the performance of faculty is being evaluated. First, the actual observed data for 89 faculties were collected. Then, using the generalized mean operator equation with different p values, the aggregated performance index for all the faculties are obtained. The results are listed in Table 6. In actual practice, the faculty is evaluated based on five different grades at Yuan-Ze. The proportion of the five grades are: $A = 10\%$, $B+ = 20\%$, $B = 40\%$, $B- = 20\%$, and $C = 10\%$. We can experimentally adjust the p value so that the rating distribution satisfies this grade requirement. The result we obtain is $p = 4.56$.

Table 4. Prior and final weights for Hierarchy 2.

Criteria	Prior weight	Final weight
quality of performance	0.6	0.661
quality of teaching	0.4	0.423
content	0.3	0.340
attitude	0.3	0.340
method	0.4	0.320
quality of research	0.4	0.384
quality of journal paper	0.2	0.246
quality of conference paper	0.2	0.153
quality of academic project	0.2	0.182
quality of practical project	0.2	0.141
quality of award	0.2	0.279
quality of service	0.2	0.193
quality of department committee	0.1	0.117
quality of internship consulting	0.2	0.166
quality of student consulting	0.2	0.204
quality of school committee	0.2	0.212
quality of administrator	0.3	0.301
quantity of performance	0.4	0.339
quantity of teaching	0.4	0.349
loading	1.0	1.000
quantity of research	0.4	0.384
quantity of journal paper	0.2	0.228
quantity of conference paper	0.2	0.173
quantity of academic project	0.2	0.209
quantity of practical project	0.2	0.192
quantity of award	0.2	0.198
quantity of service	0.2	0.216
quantity of department committee	0.1	0.139
quantity of internship consulting	0.2	0.192
quantity of student consulting	0.2	0.192
quantity of school committee	0.2	0.250
quantity of administrator	0.3	0.227

Table 5. Aggregated performance with different p for Example 1.

	$p=-\infty$	$p=-1$	$p=0$	-1	$p=+\infty$
x_1	0.2	0.347	0.408	0.457	0.8
x_2	0.2	0.426	0.476	0.521	0.8
x_3	0.2	0.393	0.457	0.516	0.8
x_4	0.2	0.389	0.456	0.521	0.8
x_5	0.2	0.362	0.408	0.455	0.8

DISCUSSIONS

The proposed approach has several advantages. The principle advantage is its simplicity and no pairwise comparison is required. Due to the analysis and comparison of different structures for the same problem, the approach also gives added insights to the problem concerned.

The approach is especially suited for large complex problems where different hierarchy structures occur frequently and naturally. In a sense, due to this natural occurrence, there is no added

Table 6. Performance of faculties with different p for Example 3.

Grade	Score	$p = -\infty$	$p = -1$	$p = 0$	$p = 1$	$p = 4.56$	$p = +\infty$
A	1.0~.8	0	4	5	7	9	28
B+	.8~.6	15	16	16	17	19	30
B	.6~.4	13	26	29	31	35	26
B–	.4~.2	36	26	23	21	17	5
C	.2~0.0	26	17	15	14	9	0

work needed to analyze the additional structures. Under these circumstances, the proposed analysis seems to be a natural outcome.

The use of the rational index provides a consistent and rational test for monitoring the assigned weights for certain hierarchy formulations where consistency for the different hierarchies can be assumed. Unfortunately, the main disadvantage is also due to this consistency requirement because of the fact that the many different hierarchies frequently serve different purposes, and thus, they do not have to have the same overall performance index.

Obviously, the approach can also handle multiple persons decision problem. Each decision maker can provide a hierarchy structure. The basic assumption is that the aggregated performance for each decision maker should be consistent. Based on this consistency assumption, which is reasonable under certain conditions, the proposed HCA can be easily applied.

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